

SPLITTING VARIETIES FOR CUP PRODUCTS WITH $\mathbb{Z}/3$ -COEFFICIENTS

BRANDON BOGGE

ABSTRACT. We connect Veronese embeddings to splitting varieties of cup products. We then give an algorithm for constructing splitting varieties for cup products with \mathbb{Z}/n coefficients, with an explicit calculation for $n = 3$. An application to the automatic realization of Galois groups is given.

1. INTRODUCTION

For a functorial assignment η of a cohomology class $\eta_F \in H^*(\text{Spec } F, \mathbb{Z}/n)$ to fields F over some ground field k , a splitting variety is a scheme X over k which has F -points if and only if η_F vanishes. Let k be a number field containing an n th root of unity and F a field extension of k . In this paper, we give an algorithm for constructing splitting varieties for cup products of elements of $H^1(\text{Spec } F, \mathbb{Z}/n)$, with an explicit calculation for $n = 3$.

It has been known for a while that quotient schemes give rise to splitting varieties. In particular, fixed points give rise to versal torsors, and we can use versal torsors to construct splitting varieties [GMS03, Example 5.4] [BF03, Prop. 4.11]. We will construct an algorithm to compute such fixed points and then run it. What comes out of this run through is long and complicated, but this is not surprising – the key ingredient in the algorithm is the Veronese embedding, and this is known to be complicated for high degree and dimension. To explain it, we must first develop some notation.

Let H be the group of upper triangular 3×3 -matrices with diagonal entries all 1 and coefficients in \mathbb{Z}/n ; this is the mod n Heisenberg group. Let $a_{ij} : H \rightarrow \mathbb{Z}/(n)$ be the function taking a matrix to its (i, j) -entry. Denote by E_{ij} the matrix such that $a_{ij}(E_{ij}) = 1$ and $a_{kl}(E_{ij}) = 0$ for $k \neq l$.

Let $N \subset H$ be the subgroup $N = \text{Ker}(a_{12} : H \rightarrow \mathbb{Z}/n)$. We have a 1-dimensional representation of N via the map $\rho : N \rightarrow \text{GL}_1(k)$ defined by $g \mapsto \zeta_n^{a_{13}(g)}$, where $\zeta_n = e^{2\pi i/n}$.

We first define a representation $V \cong k^n$ of H by the induced representation $\text{Ind}_N^H \rho$. From this, we define another representation

$$\sigma = \text{Ind}_N^H \rho \times a_{12} \times a_{23} : H \rightarrow \text{GL}(V) \times \text{GL}_2(k),$$

of H , where a_{12} and a_{23} are one dimensional representations given by $g \mapsto \zeta_n^{a_{12}(g)}$ and $g \mapsto \zeta_n^{a_{23}(g)}$, respectively. Finding the fixed ring of $k[V] \otimes_k k[\alpha, \alpha^{-1}, \beta, \beta^{-1}]$ under the representation σ gives our splitting variety.

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Since E_{13} acts by $x_i \mapsto \zeta_n x_i$ and fixes $\alpha^{\pm 1}$ and $\beta^{\pm 1}$, we find that the fixed ring

$$k[x_1, \dots, x_n, \alpha, \alpha^{-1}, \beta, \beta^{-1}]^{\langle E_{13} \rangle} \cong k[\text{Sym}^n V] \otimes_k k[\alpha, \alpha^{-1}, \beta, \beta^{-1}].$$

Since the quotient $H/\langle E_{13} \rangle \cong \mathbb{Z}/n \times \mathbb{Z}/n$ acts on $k[\text{Sym}^n V]$, there is a basis $\{E_{12}, E_{23}\}$ of $\mathbb{Z}/n \times \mathbb{Z}/n$ and a $\mathbb{Z}/n \times \mathbb{Z}/n$ -action on $\text{Sym}^n V$. But E_{12} and E_{23} commute in $H/\langle E_{13} \rangle$, so we can find a simultaneous eigenbasis of $\text{Sym}^n V$ with respect to the actions of E_{12} and E_{23} . Let $N = \binom{2n-1}{n}$ denote the degree of $\text{Sym}^n V$. We will define a surjection

$$\theta : k[z_1, \dots, z_N, \alpha, \alpha^{-1}, \beta, \beta^{-1}] \rightarrow k[x_1, \dots, x_n, \alpha, \alpha^{-1}, \beta, \beta^{-1}]^H$$

by taking the z_i to these eigenvectors weighted into the $(1, 1)$ -eigenspace – since α and β have eigenvalues $(\zeta_n, 1)$ and $(1, \zeta_n)$ under the action of (E_{12}, E_{23}) , respectively, this can be achieved by multiplying by powers of α and β .

It remains to find the kernel of θ . To do this, we define a map $\pi : k[w_1, \dots, w_N] \rightarrow k[\text{Sym}^n V]$ by taking generators to distinct degree n monomials. The kernel of this map cuts out the toric ideal of the n th Veronese embedding of \mathbb{P}^{N-1} [Stu96, 14.1]. This ideal has been widely studied, and Sturmfels gives an algorithm to compute a Gröbner basis for it. This gives a basis of $\text{Sym}^n V$, and we can define a change of basis map p from this basis to the eigenbasis under the action of $\mathbb{Z}/n \times \mathbb{Z}/n$.

We prove (Proposition 2.4) that one can take a known generating set for the toric ideal of the n th Veronese embedding of \mathbb{P}^{N-1} and obtain the kernel of θ . The proof of this does not depend on n being 3, thus giving a way of algorithmically computing splitting varieties of cup products with coefficients in \mathbb{Z}/n .

Finally, we will prove that this construction indeed produces a splitting variety. In order to state this result, we need a few more definitions. Let $\kappa : F^* \rightarrow H^1(\text{Spec } F, \mathbb{Z}/3)$ be the Kummer map, i.e. the map obtained by applying $H^*(\text{Spec } F, -)$ to the short exact sequence

$$1 \rightarrow \mu_3 \rightarrow \mathbb{G}_m \xrightarrow{z \mapsto z^3} \mathbb{G}_m \rightarrow 1$$

and identifying $\mu_3 \cong \mathbb{Z}/3$. Let $a, b \in F^*$. We perform a lengthy computation leading to the main theorem, and so refer to the result of that computation rather than restating it: let $X(a, b)$ be defined as in Definition 2.8.

Theorem 1.1. *The scheme $X(a, b)$ has an F -point if and only if $\kappa(a) \smile \kappa(b) = 0$ in $H^2(\text{Spec } F, \mathbb{Z}/3)$.*

Theorem 1.1 gives the following automatic realization result for Galois groups, which we will discuss in Section 3.

Theorem 1.2. *Suppose $a, b \in F^*$ are such that $F(\sqrt[3]{a}, \sqrt[3]{b})/F$ is a $\mathbb{Z}/3 \times \mathbb{Z}/3$ -Galois extension. Then the following are equivalent:*

- (1) *There exists a $\mathbb{Z}/3$ -Galois extension $L/F(\sqrt[3]{a}, \sqrt[3]{b})$ such that L/F is an H -Galois extension;*
- (2) *$\kappa(a) \smile \kappa(b) = 0$ in $H^2(\text{Spec } F, \mathbb{Z}/3)$;*
- (3) *The scheme $X(a, b)$ has an F -point.*

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2. SPLITTING VARIETY

Let H be the group of upper triangular 3×3 -matrices with diagonal entries all 1 and coefficients in $\mathbb{Z}/3$; this is the mod 3 Heisenberg group. Let $a_{ij} : H \rightarrow \mathbb{Z}/(3)$ be the function taking a matrix to its (i, j) -entry. Denote by E_{ij} the matrix such that $a_{ij}(E_{ij}) = 1$ and $a_{kl}(E_{ij}) = 0$ for $k \neq l$.

Let $N \subset H$ be the subgroup $N = \text{Ker}(a_{12} : H \rightarrow \mathbb{Z}/3)$. We have a 1-dimensional representation of N via the map $\rho : N \rightarrow \text{GL}_1(k)$ defined by $g \mapsto \zeta^{a_{13}(g)}$, where $\zeta = e^{2\pi i/3}$.

We get an induced representation $\text{Ind}_N^H \rho$ of H . Following the procedure in [FH91, pp. 32-33], we can find a homomorphism $H \rightarrow \text{GL}(V) \cong \text{GL}_3(k)$ corresponding to $\text{Ind}_N^H \rho$. Given a basis $\{u_0, u_1, u_2\}$ of V and $g \in H$, this representation acts on V by

$$gu_i = \zeta^{a_{13}(g) - (i + a_{12}(g))a_{13}(g)} u_{i + a_{12}(g) \bmod 3}.$$

Let

$$(1) \quad \sigma = \text{Ind}_N^H \rho \times a_{12} \times a_{23} : H \rightarrow \text{GL}(V) \times \text{GL}_2(k),$$

where a_{12} and a_{23} are one dimensional representations given by $g \mapsto \zeta^{a_{12}(g)}$ and $g \mapsto \zeta^{a_{23}(g)}$, respectively. This gives another representation of H .

Let

$$k[x_1, \dots, x_n]^H = \{f \in k[x_1, \dots, x_n] : gf = f \quad \forall g \in H\}$$

We would like to express $k[x_1, x_2, x_3, \alpha, \alpha^{-1}, \beta, \beta^{-1}]^H$ as a quotient $k[z_1, \dots, z_N]/\langle f_1, \dots, f_m \rangle$ for the representation σ of (1). Define a map

$$\theta : k[z_1, \dots, z_{10}, a, a^{-1}, b, b^{-1}] \rightarrow k[x_1, x_2, x_3, \alpha, \alpha^{-1}, \beta, \beta^{-1}]$$

by

$$(2) \quad \begin{array}{lll} z_1 \mapsto x_1^3 + x_2^3 + x_3^3 & z_6 \mapsto \alpha\beta^2(\zeta x_1^2 x_3 + \zeta^2 x_2^2 x_1 + x_3^2 x_2) & a \mapsto \alpha^3 \\ z_2 \mapsto \alpha^2(\zeta^2 x_1^3 + \zeta x_2^3 + x_3^3) & z_7 \mapsto \beta(x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_1) & a^{-1} \mapsto \alpha^{-3} \\ z_3 \mapsto \alpha(\zeta x_1^3 + \zeta^2 x_2^3 + x_3^3) & z_8 \mapsto \alpha^2\beta(\zeta^2 x_1^2 x_2 + \zeta x_2^2 x_3 + x_3^2 x_1) & b \mapsto \beta^3 \\ z_4 \mapsto \beta^2(x_1^2 x_3 + x_2^2 x_1 + x_3^2 x_2) & z_9 \mapsto \alpha\beta(\zeta x_1^2 x_2 + \zeta^2 x_2^2 x_3 + x_3^2 x_1) & b^{-1} \mapsto \beta^{-3} \\ z_5 \mapsto \alpha^2\beta^2(\zeta^2 x_1^2 x_3 + \zeta x_2^2 x_1 + x_3^2 x_2) & z_{10} \mapsto x_1 x_2 x_3 & \end{array}$$

The following proposition gives an explanation for this map.

Proposition 2.1. *The image of θ is $k[x_1, x_2, x_3, \alpha, \alpha^{-1}, \beta, \beta^{-1}]^H$.*

Proof. The action of E_{13} takes $E_{13}(x_i) = \zeta x_i$ for $i = 1, 2, 3$ and fixes $\alpha^{\pm 1}$ and $\beta^{\pm 1}$. Thus

$$k[x_1, x_2, x_3, \alpha, \alpha^{-1}, \beta, \beta^{-1}]^{\langle E_{13} \rangle} \cong k[\text{Sym}^3 V] \otimes_k k[\alpha, \alpha^{-1}, \beta, \beta^{-1}]$$

Now, $H/\langle E_{13} \rangle \cong \mathbb{Z}/3 \times \mathbb{Z}/3$ has a $\mathbb{Z}/3$ basis $\{E_{12}, E_{23}\}$. The action of E_{12} permutes the x_i by $(x_1 \ x_2 \ x_3)$ (here we use cycle notation, cf. [Art10, § 1.5]) and takes $E_{12}(\alpha^{\pm 1}) = \zeta \alpha^{\pm 1}$, $E_{12}(\beta^{\pm 1}) = \beta^{\pm 1}$. The action of E_{23} is given by $E_{23}(x_1) = x_1$, $E_{23}(x_2) = \zeta^2 x_2$, $E_{23}(x_3) = \zeta x_3$, $E_{23}(\alpha^{\pm 1}) = \alpha^{\pm 1}$, $E_{23}(\beta^{\pm 1}) = \beta^{\pm 1}$. Hence there is a simultaneous eigenbasis for $\mathbb{Z}/3 \times \mathbb{Z}/3$ of $\text{Sym}^3 V \oplus k\alpha \oplus k\alpha^{-1} \oplus k\beta \oplus k\beta^{-1}$ given by

	Eigenvector	Eigenvalues
	$x_1^2 x_3 + x_2^2 x_1 + x_3^2 x_2$	$(1, \zeta)$
	$x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_1$	$(1, \zeta^2)$
	$x_1^3 + x_2^3 + x_3^3$	$(1, 1)$
	$x_1 x_2 x_3$	$(1, 1)$
	$\zeta^2 x_1^2 x_3 + \zeta x_2^2 x_1 + x_3^2 x_2$	(ζ, ζ)
	$\zeta^2 x_1^2 x_2 + \zeta x_2^2 x_3 + x_3^2 x_1$	(ζ, ζ^2)
	$\zeta^2 x_1^3 + \zeta x_2^3 + x_3^3$	$(\zeta, 1)$
	$\zeta x_1^2 x_3 + \zeta^2 x_2^2 x_1 + x_3^2 x_2$	(ζ^2, ζ)
	$\zeta x_1^2 x_2 + \zeta^2 x_2^2 x_3 + x_3^2 x_1$	(ζ^2, ζ^2)
	$\zeta x_1^3 + \zeta^2 x_2^3 + x_3^3$	$(\zeta^2, 1)$
	α	$(\zeta, 1)$
	α^{-1}	$(\zeta^2, 1)$
	β	$(1, \zeta)$
	β^{-1}	$(1, \zeta^2)$

Label these eigenvectors as v_1, \dots, v_{14} . Then any element $f \in k[\text{Sym}^3 V] \otimes_k k[\alpha, \alpha^{-1}, \beta, \beta^{-1}]$ can be expressed as a polynomial in the v_i . Since each monomial of f in the variables v_i, α, β goes to a scalar multiple of itself under the action of any element in H , f is fixed by H if and only if each monomial is. Therefore $k[x_1, x_2, x_3, \alpha, \alpha^{-1}, \beta, \beta^{-1}]^H$ is generated by products of the eigenvectors whose eigenvalues under E_{12} and E_{23} multiply to 1, which we claim is the image of θ .

Let $(\zeta^{m_i}, \zeta^{n_i})$ be the eigenvalues of v_i under (E_{12}, E_{23}) , respectively. Then, re-indexing the v_i as necessary,

$$\theta(z_i) = \alpha^{3-m_i} \beta^{3-n_i} v_i$$

by (2) (this even works for $\alpha^{\pm 1}$ and $\beta^{\pm 1}$, considering z_{11} through z_{14} to be a, a^{-1}, b, b^{-1} , respectively). If the monomial $v_1^{r_1} \cdots v_{14}^{r_{14}}$ is fixed, then $\sum r_i m_i$ and $\sum r_i n_i$ are both divisible by 3, so that the eigenvalue of the product is $(1, 1)$. Hence

$$\prod_{i=1}^{14} \theta(z_i^{r_i}) = \prod_{i=1}^{14} \alpha^{r_i(3-m_i)} \beta^{r_i(3-n_i)} v_i^{r_i} = \alpha^{3m} \beta^{3n} \prod_{i=1}^{14} v_i^{r_i}.$$

for some $m, n \in \mathbb{Z}$. Therefore

$$\prod_{i=1}^{14} v_i^{r_i} = \theta(a^{-m}) \theta(b^{-n}) \prod_{i=1}^{14} \theta(z_i^{r_i}) \in \text{Im}(\theta).$$

□

The eigenspace decomposition of $k[x_1, x_2, x_3, \alpha, \alpha^{-1}, \beta, \beta^{-1}]^{\langle E_{13} \rangle}$ means that any degree 3 monomial in $k[x_1, x_2, x_3]$ can be expressed in terms of the eigenvectors for $\mathbb{Z}/3 \times \mathbb{Z}/3$. This will be useful in finding the kernel of θ .

Define the map $\pi : k[w_1, \dots, w_{10}] \rightarrow k[\text{Sym}^3 V]$ by $w_1 \mapsto x_1 x_2 x_3, w_2 \mapsto x_1^2$, etc. The kernel of π cuts out the toric ideal of the 3rd Veronese embedding of \mathbb{P}^2 [Stu96], and Macaulay2 [GS] can find the generators:

(4)

$$\begin{aligned} \text{Ker } \pi = \langle & w_8 w_9 - w_1 w_{10}, w_7 w_9 - w_5 w_{10}, w_5 w_9 - w_2 w_{10}, w_3 w_9 - w_7 w_{10}, w_1 w_9 - w_6 w_{10}, \\ & w_8^2 - w_3 w_{10}, w_6 w_8 - w_5 w_{10}, w_4 w_8 - w_{10}^2, w_1 w_8 - w_7 w_{10}, w_6 w_7 - w_2 w_8, \\ & w_4 w_7 - w_1 w_{10}, w_1 w_7 - w_5 w_8, w_6^2 - w_2 w_9, w_4 w_6 - w_9^2, w_3 w_6 - w_5 w_8, \\ & w_1 w_6 - w_2 w_{10}, w_5^2 - w_2 w_7, w_4 w_5 - w_6 w_{10}, w_3 w_5 - w_7^2, w_1 w_5 - w_2 w_8, \\ & w_3 w_4 - w_8 w_{10}, w_2 w_4 - w_6 w_9, w_1 w_4 - w_9 w_{10}, w_2 w_3 - w_5 w_7, w_1 w_3 - w_7 w_8, \\ & w_1 w_2 - w_5 w_6, w_1^2 - w_5 w_{10} \rangle. \end{aligned}$$

There are 27 generators, each of which falls into one of five categories:

- 1) Relations of the form $x_i^2 x_j \cdot x_j^2 x_i = x_i^3 \cdot x_j^3$. There are three such relations. These are mapped to by $w_5 w_7 - w_2 w_3, w_6 w_9 - w_2 w_4$, and $w_8 w_{10} - w_3 w_4$.
- 2) Relations of the form $x_i^2 x_j \cdot x_i^2 x_k = x_i^3 \cdot x_i x_j x_k$. There are three of these relations. These are mapped to by $w_5 w_6 - w_1 w_2, w_7 w_8 - w_1 w_3$, and $w_9 w_{10} - w_1 w_4$.
- 3) Relations of the form $(x_i^2 x_j)^2 = x_i^3 \cdot x_j^2 x_i$. There are six of these, and they are mapped to by $w_5^2 - w_2 w_7, w_6^2 - w_2 w_9, w_7^2 - w_3 w_5, w_8^2 - w_3 w_{10}, w_9^2 - w_4 w_6$, and $w_{10}^2 - w_4 w_8$.
- 4) Relations of the form $x_i^2 x_j \cdot x_j^2 x_k = x_j^3 \cdot x_i^2 x_k = x_j^2 x_i \cdot x_i x_j x_k$. There are six products of this form, each of which gives two relations; thus there are 12 relations of this form.
- 5) There are three ways to write $(x_i x_j x_k)^2$ in terms of the other monomials. These are mapped to by $w_1^2 - w_5 w_{10}, w_1^2 - w_6 w_8$, and $w_1^2 - w_7 w_9$.

These give exactly all 27 generators for $\text{Ker } \pi$.

Remark 2.2. Extending π to the map

$$\pi \otimes \mathbb{1} : k[w_1, \dots, w_{10}] \otimes_k k[\alpha, \alpha^{-1}, \beta, \beta^{-1}] \rightarrow k[\text{Sym}^3 V] \otimes_k k[\alpha, \alpha^{-1}, \beta, \beta^{-1}]$$

does not affect the kernel because $k[\alpha, \alpha^{-1}, \beta, \beta^{-1}]$ is a flat k -algebra.

As in the proof of Proposition 2.1, we label the eigenvectors of $\text{Sym}^3 V$ as v_1, \dots, v_{10} in such a way that $\theta(z_i)$ is a multiple of v_i . Let $\pi' : k[v_1, \dots, v_{10}] \rightarrow k[\text{Sym}^3 V]$ take v_i to its expression in (3).

Since H acts on V , the quotient $H/\langle E_{13} \rangle$ acts on the fixed elements $k[V]^{\langle E_{13} \rangle} \cong k[\text{Sym}^3 V]$, so there is a $\mathbb{Z}/3 \times \mathbb{Z}/3$ action on $k[\text{Sym}^3 V]$ with basis $\{E_{12}, E_{23}\}$. This gives a natural action on $k[v_1, \dots, v_{10}]$. For any $g \in \mathbb{Z}/3 \times \mathbb{Z}/3$, we have that $g\pi'(v_i) = \zeta^{k_i}\pi'(v_i)$ for some k_i . We then define

$$gv_i := \zeta^{k_i} v_i.$$

It follows that π' is equivariant.

Denote by p the map $k[w_1, \dots, w_{10}] \rightarrow k[v_1, \dots, v_{10}]$ resulting from the change of basis corresponding to the two bases of $\text{Sym}^3 V$ given by $\{w_1, \dots, w_{10}\}$ and $\{v_1, \dots, v_{10}\}$. Note that

$$\begin{array}{ccc} k[w_1, \dots, w_{10}] & \xrightarrow{p} & k[v_1, \dots, v_{10}] \\ & \searrow \pi & \downarrow \pi' \\ & & k[\text{Sym}^3 V] \end{array}$$

commutes and that p is an isomorphism. We want a map $\varphi : k[z_1, \dots, z_{10}, a, a^{-1}, b, b^{-1}] \rightarrow k[w_1, \dots, w_{10}, \alpha, \alpha^{-1}, \beta, \beta^{-1}]$ such that tensoring with $k[\alpha, \alpha^{-1}, \beta, \beta^{-1}]$ makes

(5)

$$\begin{array}{ccccc} k[z_1, \dots, z_{10}, \alpha, \alpha^{-1}, \beta, \beta^{-1}] & \xrightarrow{\varphi} & k[w_1, \dots, w_{10}, \alpha, \alpha^{-1}, \beta, \beta^{-1}] & \xrightarrow{p \otimes 1} & k[v_1, \dots, v_{10}, \alpha, \alpha^{-1}, \beta, \beta^{-1}] \\ & \searrow \theta & \downarrow \pi \otimes 1 & \swarrow \pi' \otimes 1 & \\ & & k[\text{Sym}^3 V] \otimes_k k[\alpha, \alpha^{-1}, \beta, \beta^{-1}] & & \\ & & \downarrow & & \\ & & k[V] \otimes_k k[\alpha, \alpha^{-1}, \beta, \beta^{-1}] & & \end{array}$$

commute. This map is given by

$$\begin{array}{lll} (6) \quad z_1 \mapsto w_2 + w_3 + w_4 & z_6 \mapsto \alpha\beta^2(\zeta w_6 + \zeta^2 w_7 + w_{10}) & a \mapsto \alpha^3 \\ z_2 \mapsto \alpha^2(\zeta^2 w_2 + \zeta w_3 + w_4) & z_7 \mapsto \beta(w_5 + w_8 + w_9) & a^{-1} \mapsto \alpha^{-3} \\ z_3 \mapsto \alpha(\zeta w_2 + \zeta^2 w_3 + w_4) & z_8 \mapsto \alpha^2\beta(\zeta^2 w_5 + \zeta w_8 + w_9) & b \mapsto \beta^3 \\ z_4 \mapsto \beta^2(w_6 + w_7 + w_{10}) & z_9 \mapsto \alpha\beta(\zeta w_5 + \zeta^2 w_8 + w_9) & b^{-1} \mapsto \beta^{-3} \\ z_5 \mapsto \alpha^2\beta^2(\zeta^2 w_6 + \zeta w_7 + w_{10}) & z_{10} \mapsto w_1 & \end{array}$$

as can be verified by comparing (6) to (2). The map φ is used to bring results about Veronese embeddings into the picture, and p allows us to translate this information into statements about eigenspaces under $\mathbb{Z}/3 \times \mathbb{Z}/3$.

Given a generator $w_i w_j - w_k w_l$ of $\text{Ker } \pi$, we can construct an element of $\text{Ker } \theta$ as follows: let $h = p(w_i w_j - w_k w_l)$. Observe that $\pi'(h) = 0$. Each monomial in h belongs to some eigenspace; let $h_{m,n}$ be the projection of h onto the (ζ^m, ζ^n) eigenspace, so that

$$h = \sum_{0 \leq m, n < 3} h_{m,n}.$$

Because π' is equivariant, the $\pi'(h_{m,n})$ are in distinct eigenspaces. It follows that $\pi'(h) = 0$ if and only if each $\pi'(h_{m,n}) = 0$. By construction, each $\alpha^{-m}\beta^{-n}\pi'(h_{m,n}) \in \text{Im } \theta$, with the exponents of α and β taken mod 3. Thus if $\pi'(h) = 0$, the elements $\alpha^{-m}\beta^{-n}\pi'(h_{m,n})$ are in $\text{Ker } \theta$.

Example 2.3. We will construct elements of $\text{Ker } \theta$ from $w_8 w_9 - w_1 w_{10} \in \text{Ker } \pi$ using this method. We first apply p to get

$$h = \frac{1}{9}(v_7 + \zeta^2 v_8 + \zeta v_9)(v_7 + v_8 + v_9) - \frac{1}{3}v_{10}(v_4 + v_5 + v_6) \in k[v_1, \dots, v_{10}]$$

We get the decomposition into eigenspaces (ignoring those with no contribution)

$$\begin{aligned} h_{0,1} &= v_7^2 - v_8 v_9 - 3v_4 v_{10} \\ h_{1,1} &= \zeta v_9^2 - \zeta v_7 v_8 - 3v_5 v_{10} \\ h_{2,1} &= \zeta^2 v_8^2 - \zeta^2 v_7 v_9 - 3v_6 v_{10}. \end{aligned}$$

We then see from (2) that

$$\begin{aligned} \theta \left(z_7^2 - \frac{1}{a} z_8 z_9 - 3z_4 z_{10} \right) &= \beta^2 \pi'(h_{0,1}) \\ \theta \left(\zeta z_9^2 - \zeta z_7 z_8 - 3z_5 z_{10} \right) &= \alpha^2 \beta^2 \pi'(h_{1,1}) \\ \theta \left(\frac{\zeta^2}{a} z_8^2 - \zeta^2 z_7 z_9 - 3z_6 z_{10} \right) &= \alpha \beta^2 \pi'(h_{2,1}), \end{aligned}$$

and it is straightforward to verify that these elements are indeed in $\text{Ker } \theta$.

We now show that all elements of $\text{Ker } \theta$ come about from this process.

Proposition 2.4. *Given $\langle u_1, \dots, u_M \rangle$ a generating set for $\text{Ker } \pi$, we can find a set of generators for $\text{Ker } \theta$.*

Proof. Since the maps φ and $p \otimes \mathbb{1}$ are inejctive, we can view $k[z_1, \dots, z_{10}, \alpha, \alpha^{-1}, \beta, \beta^{-1}]$ as a subring of $k[v_1, \dots, v_{10}, \alpha, \alpha^{-1}, \beta, \beta^{-1}]$. By Proposition 2.1, this subring is the $(1, 1)$ -eigenspace. In view of (5), finding $\text{Ker}(\theta)$ can be reduced to finding the intersection of $\text{Ker}(\pi' \otimes \mathbb{1})$ with the $(1, 1)$ -eigenspace of $k[v_1, \dots, v_{10}, \alpha, \alpha^{-1}, \beta, \beta^{-1}]$.

Given generators for $\text{Ker}(\pi)$, we can obtain generators for $\text{Ker}(\pi' \otimes \mathbb{1})$ as follows. By Remark 2.2, we then know that $\text{Ker}(\pi \otimes \mathbb{1}) = \text{Ker}(\pi) \otimes_k k[\alpha, \alpha^{-1}, \beta, \beta^{-1}]$. Then, since $p \otimes \mathbb{1}$ is an isomorphism, we find that $\text{Ker}(\pi' \otimes \mathbb{1}) = p \otimes \mathbb{1}(\text{Ker}(\pi \otimes \mathbb{1}))$. In particular, given finitely many generators u_1, \dots, u_M of $\text{Ker } \pi$, we get that $\text{Ker}(\pi' \otimes \mathbb{1}) = \langle p(u_1) \otimes \mathbb{1}, \dots, p(u_M) \otimes \mathbb{1} \rangle$.

Let h_1, \dots, h_N denote the projections onto eigenspaces of the generators of $\mathfrak{a} = \text{Ker}(\pi' \otimes \mathbb{1})$. Note that the ring $k[v_1, \dots, v_{10}, \alpha, \alpha^{-1}, \beta, \beta^{-1}]$ is a direct sum of eigenspaces under the action of $\mathbb{Z}/3 \times \mathbb{Z}/3$. That $\mathfrak{a} \subset \langle h_1, \dots, h_N \rangle$ is immediate, and the other direction follows because π' is equivariant. Indeed, if $h \in \mathfrak{a}$ and $h_{m,n}$ is its projection onto the (ζ^m, ζ^n) -eigenspace, then

$$\sum_{0 \leq m, n < 3} \pi' \otimes \mathbb{1}(h_{m,n}) = 0.$$

But each of these terms is in a distinct eigenspace because π' is equivariant, and so each term must itself be 0.

Let h_i be in the $(\zeta^{m_i}, \zeta^{n_i})$ -eigenspace, so that $\alpha^{-m_i} \beta^{-n_i} h_i$ is in the $(1, 1)$ -eigenspace. We claim that $\langle \alpha^{-m_1} \beta^{-n_1} h_1, \dots, \alpha^{-m_N} \beta^{-n_N} h_N \rangle$ is the intersection of \mathfrak{a} with the $(1, 1)$ -eigenspace, which we will denote by $\mathfrak{a}_{(1,1)}$. The inclusion $\langle \alpha^{-m_1} \beta^{-n_1} h_1, \dots, \alpha^{-m_N} \beta^{-n_N} h_N \rangle \subset \mathfrak{a}_{(1,1)}$ follows immediately. Now let $f \in \mathfrak{a}_{(1,1)}$. Then we can write

$$f = \sum_{i=1}^N g_i h_i$$

where each $g_i \in k[v_1, \dots, v_{10}, \alpha, \alpha^{-1}, \beta, \beta^{-1}]$. In order for f to be in the $(1, 1)$ -eigenspace, all of the terms $g_i h_i$ in a different eigenspace must cancel out; hence we may assume that each $g_i h_i$ is in the $(1, 1)$ -eigenspace. By assumption, h_i is in the $(\zeta^{m_i}, \zeta^{n_i})$ -eigenspace, so g_i is in the $(\zeta^{-m_i}, \zeta^{-n_i})$ -eigenspace. Let $g'_i = \alpha^{m_i} \beta^{n_i} g_i$. Then g'_i is in the $(1, 1)$ -eigenspace, and

$$f = \sum_i g'_i (\alpha^{-m_i} \beta^{-n_i} h_i).$$

□

Remark 2.5. Given generators for the toric ideal of the appropriate Veronese embedding, Proposition 2.4 gives a way to algorithmically compute splitting varieties for cup products with coefficients in \mathbb{Z}/n for any n .

Performing this calculation for each generator in (4) yields

Proposition 2.6. *The kernel of θ is generated by*

- $z_7^2 - \frac{1}{a} z_8 z_9 - 3 z_4 z_{10}$
- $\frac{\zeta^2}{a} z_8^2 - \zeta^2 z_7 z_9 - 3 z_6 z_{10}$
- $\zeta z_9^2 - \zeta z_7 z_8 - 3 z_5 z_{10}$
- $(1 - \zeta) z_4 z_8 + (\zeta^2 - 1) z_5 z_7 + (\zeta - \zeta^2) z_6 z_9$
- $(1 - \zeta^2) z_4 z_9 + \frac{\zeta^2 - \zeta}{a} z_5 z_8 + (\zeta - 1) z_6 z_7$
- $z_7^2 - \frac{1}{a} z_8 z_9 - z_1 z_4 - \frac{\zeta}{a} z_2 z_6 - \frac{\zeta^2}{a} z_3 z_5$
- $\frac{\zeta}{a} z_8^2 - \zeta z_7 z_9 - z_1 z_6 - \frac{\zeta}{a} z_2 z_5 - \zeta^2 z_3 z_4$
- $\zeta^2 z_9^2 - \zeta^2 z_7 z_8 - z_1 z_5 - \zeta z_2 z_4 - \zeta^2 z_3 z_6$
- $z_1 z_7 + \frac{\zeta}{a} z_2 z_9 + \frac{\zeta}{a} z_3 z_8 - \frac{1}{b} z_4^2 + \frac{1}{ab} z_5 z_6$
- $z_1 z_8 + \zeta^2 z_2 z_7 + \zeta z_3 z_9 - \frac{\zeta}{b} z_6^2 + \frac{\zeta}{b} z_4 z_5$
- $z_1 z_9 + \frac{\zeta^2}{a} z_2 z_8 + \zeta z_3 z_7 - \frac{\zeta^2}{ab} z_5^2 + \frac{\zeta^2}{b} z_4 z_6$
- $\frac{1}{b} z_4^2 - \frac{1}{ab} z_5 z_6 - 3 z_7 z_{10}$
- $\frac{\zeta}{ab} z_5^2 - \frac{\zeta}{b} z_4 z_6 - 3 z_9 z_{10}$
- $\frac{\zeta^2}{b} z_6^2 - \frac{\zeta^2}{b} z_4 z_5 - 3 z_8 z_{10}$
- $z_7^2 + \frac{2}{a} z_8 z_9 - z_1 z_4 - \frac{\zeta^2}{a} z_2 z_6 - \frac{\zeta}{a} z_3 z_5$
- $\frac{\zeta}{a} z_8^2 + 2 \zeta z_7 z_9 - z_1 z_6 - \frac{\zeta^2}{a} z_2 z_5 - \zeta z_3 z_4$
- $\zeta^2 z_9^2 + 2 \zeta^2 z_7 z_8 - z_1 z_5 - \zeta^2 z_2 z_4 - \zeta z_3 z_6$
- $(\zeta^2 - \zeta) z_4 z_8 + (\zeta - 1) z_5 z_7 + (1 - \zeta^2) z_6 z_9$
- $(\zeta - \zeta^2) z_4 z_9 + \frac{1 - \zeta}{a} z_5 z_8 + (\zeta^2 - 1) z_6 z_7$
- $z_1 z_7 + \frac{\zeta}{a} z_2 z_9 + \frac{\zeta^2}{a} z_3 z_8 - \frac{1}{b} z_4^2 - \frac{2}{ab} z_5 z_6$
- $\zeta^2 z_1 z_8 + z_2 z_7 + \zeta z_3 z_9 - \frac{1}{b} z_6^2 - \frac{2}{b} z_4 z_5$
- $\zeta z_1 z_9 + \frac{\zeta^2}{a} z_2 z_8 + z_3 z_7 - \frac{1}{ab} z_5^2 - \frac{2}{b} z_4 z_6$
- $\frac{1}{b} z_4^2 - \frac{1}{ab} z_5 z_6 - 3 z_7 z_{10}$
- $\frac{\zeta^2}{ab} z_5^2 - \frac{\zeta^2}{b} z_4 z_6 - 3 \zeta z_9 z_{10}$
- $\frac{\zeta}{b} z_6^2 - \frac{\zeta}{b} z_4 z_5 - 3 \zeta^2 z_8 z_{10}$
- $\frac{1}{b} z_4^2 - \frac{1}{ab} z_5 z_6 - z_1 z_7 - \frac{\zeta^2}{a} z_2 z_9 - \frac{\zeta}{a} z_3 z_8$
- $\frac{1}{ab} z_5^2 - \frac{1}{b} z_4 z_6 - \zeta z_1 z_9 - \frac{1}{a} z_2 z_8 - \zeta^2 z_3 z_7$

- $\frac{1}{b}z_6^2 - \frac{1}{b}z_4z_5 - \zeta^2z_1z_8 - \zeta z_2z_7 - z_3z_9$
- $z_1z_4 + \frac{\zeta}{a}z_2z_6 + \frac{\zeta^2}{a}z_3z_5 - 3z_4z_{10}$
- $\zeta^2z_1z_5 + z_2z_4 + \zeta z_3z_6 - 3z_5z_{10}$
- $\zeta z_1z_6 + \frac{\zeta^2}{a}z_2z_5 + z_3z_4 - 3z_6z_{10}$
- $z_7^2 - \frac{1}{a}z_8z_9 - 3z_4z_{10}$
- $\frac{1}{a}z_8^2 - z_7z_9 - 3\zeta z_6z_{10}$
- $z_9^2 - z_7z_8 - 3\zeta^2z_5z_{10}$
- $\frac{1}{b}z_4^2 + \frac{2}{ab}z_5z_6 - z_1z_7 - \frac{\zeta}{a}z_2z_9 - \frac{\zeta^2}{a}z_3z_8$
- $\frac{\zeta^2}{ab}z_5^2 + \frac{2\zeta^2}{b}z_4z_6 - z_1z_9 - \frac{\zeta}{a}z_2z_8 - \zeta^2z_3z_7$
- $\frac{\zeta}{b}z_6^2 + \frac{2\zeta}{b}z_4z_5 - z_1z_8 - \zeta z_2z_7 - \zeta^2z_3z_9$
- $z_1z_4 + \frac{\zeta^2}{a}z_2z_6 + \frac{\zeta}{a}z_3z_5 - z_7^2 - \frac{2}{a}z_8z_9$
- $\zeta z_1z_5 + z_2z_4 + \zeta^2z_3z_6 - z_9^2 - 2z_7z_8$
- $\zeta^2z_1z_6 + \frac{\zeta}{a}z_2z_5 + z_3z_4 - \frac{1}{a}z_8^2 - 2z_7z_9$
- $z_1z_4 + \frac{\zeta}{a}z_2z_6 + \frac{\zeta^2}{a}z_3z_5 - z_7^2 + \frac{1}{a}z_8z_9$
- $\zeta z_1z_5 + \zeta^2z_2z_4 + z_3z_6 - z_9^2 + z_7z_8$
- $\zeta^2z_1z_6 + \frac{1}{a}z_2z_5 + \zeta z_3z_4 - \frac{1}{a}z_8^2 + z_7z_9$
- $z_1z_4 + \frac{\zeta}{a}z_2z_6 + \frac{\zeta^2}{a}z_3z_5 - 3z_4z_{10}$
- $z_1z_5 + \zeta z_2z_4 + \zeta^2z_3z_6 - 3\zeta z_5z_{10}$
- $z_1z_6 + \frac{\zeta}{a}z_2z_5 + \zeta^2z_3z_4 - 3\zeta^2z_6z_{10}$
- $z_7^2 + \frac{2}{a}z_8z_9 - z_1z_4 - \frac{\zeta^2}{a}z_2z_6 - \frac{\zeta}{a}z_3z_5$
- $\frac{\zeta^2}{a}z_8^2 + 2\zeta^2z_7z_9 - \zeta z_1z_6 - \frac{1}{a}z_2z_5 - \zeta^2z_3z_4$
- $\zeta z_9^2 + 2\zeta z_7z_8 - \zeta^2z_1z_5 - \zeta z_2z_4 - z_3z_6$
- $z_1z_7 + \frac{\zeta^2}{a}z_2z_9 + \frac{\zeta}{a}z_3z_8 - \frac{1}{b}z_4^2 + \frac{1}{ab}z_5z_6$
- $\zeta z_1z_8 + z_2z_7 + \zeta^2z_3z_9 - \frac{\zeta^2}{b}z_6^2 + \frac{\zeta^2}{b}z_4z_5$
- $\zeta^2z_1z_9 + \frac{\zeta}{a}z_2z_8 + z_3z_7 - \frac{\zeta}{ab}z_5^2 + \frac{\zeta}{b}z_4z_6$
- $z_1z_7 + \frac{\zeta}{a}z_2z_9 + \frac{\zeta^2}{a}z_3z_8 - \frac{1}{b}z_4^2 - \frac{2}{ab}z_5z_6$
- $\zeta z_1z_8 + \zeta^2z_2z_7 + z_3z_9 - \frac{\zeta^2}{b}z_6^2 - \frac{2\zeta^2}{b}z_4z_5$
- $\zeta^2z_1z_9 + \frac{1}{a}z_2z_8 + \zeta z_3z_7 - \frac{\zeta}{ab}z_5^2 - \frac{2\zeta}{b}z_4z_6$
- $z_1z_7 + \frac{\zeta^2}{a}z_2z_9 + \frac{\zeta}{a}z_3z_8 - 3z_7z_{10}$
- $\zeta^2z_1z_8 + \zeta z_2z_7 + z_3z_9 - 3\zeta z_8z_{10}$
- $\zeta z_1z_9 + \frac{1}{a}z_2z_8 + \zeta^2z_3z_7 - 3\zeta^2z_9z_{10}$
- $z_1^2 - \frac{1}{a}z_2z_3 - \frac{1}{b}z_4z_7 - \frac{\zeta}{ab}z_5z_9 - \frac{\zeta^2}{ab}z_6z_8$
- $\frac{\zeta^2}{a}z_2^2 - \zeta^2z_1z_3 - \frac{\zeta}{b}z_4z_9 - \frac{\zeta^2}{ab}z_5z_8 - \frac{1}{b}z_6z_7$
- $\zeta z_3^2 - \zeta z_1z_2 - \frac{\zeta^2}{b}z_4z_8 - \frac{1}{b}z_5z_7 - \frac{\zeta}{b}z_6z_9$
- $z_1^2 - \frac{1}{a}z_2z_3 - \frac{1}{b}z_4z_7 - \frac{\zeta}{ab}z_5z_9 - \frac{\zeta^2}{ab}z_6z_8$
- $\frac{\zeta}{a}z_2^2 - \zeta z_1z_3 - \frac{1}{b}z_4z_9 - \frac{\zeta}{ab}z_5z_8 - \frac{\zeta^2}{b}z_6z_7$
- $\zeta^2z_3^2 - \zeta^2z_1z_2 - \frac{1}{b}z_4z_8 - \frac{\zeta}{b}z_5z_7 - \frac{\zeta^2}{b}z_6z_9$
- $\frac{1}{b}z_4z_7 + \frac{1}{ab}z_5z_9 + \frac{1}{ab}z_6z_8 - 3z_1z_{10}$
- $\frac{1}{b}z_4z_8 + \frac{1}{b}z_5z_7 + \frac{1}{b}z_6z_9 - 3z_2z_{10}$
- $\frac{1}{b}z_4z_9 + \frac{1}{ab}z_5z_8 + \frac{1}{b}z_6z_7 - 3z_3z_{10}$
- $z_1^2 - \frac{1}{a}z_2z_3 - \frac{1}{b}z_4z_7 - \frac{\zeta}{ab}z_5z_9 - \frac{\zeta^2}{ab}z_6z_8$
- $\frac{1}{a}z_2^2 - z_1z_3 - \frac{\zeta^2}{b}z_4z_9 - \frac{1}{ab}z_5z_8 - \frac{\zeta}{b}z_6z_7$

- $z_3^2 - z_1 z_2 - \frac{\zeta}{b} z_4 z_8 - \frac{\zeta^2}{b} z_5 z_7 - \frac{1}{b} z_6 z_9$
- $\frac{1}{b} z_4 z_7 + \frac{1}{ab} z_5 z_9 + \frac{1}{ab} z_6 z_8 - 3 z_1 z_{10}$
- $\frac{\zeta^2}{b} z_4 z_8 + \frac{\zeta^2}{b} z_5 z_7 + \frac{\zeta^2}{b} z_6 z_9 - 3 \zeta^2 z_2 z_{10}$
- $\frac{\zeta}{b} z_4 z_9 + \frac{\zeta}{ab} z_5 z_8 + \frac{\zeta}{b} z_6 z_7 - 3 \zeta z_3 z_{10}$
- $\frac{1}{b} z_4 z_7 + \frac{1}{ab} z_5 z_9 + \frac{1}{ab} z_6 z_8 - 3 z_1 z_{10}$
- $\frac{\zeta}{b} z_4 z_8 + \frac{\zeta}{b} z_5 z_7 + \frac{\zeta}{b} z_6 z_9 - 3 \zeta z_2 z_{10}$
- $\frac{\zeta^2}{b} z_4 z_9 + \frac{\zeta^2}{ab} z_5 z_8 + \frac{\zeta^2}{b} z_6 z_7 - 3 \zeta^2 z_3 z_{10}$
- $\frac{1}{b} z_4 z_7 + \frac{\zeta^2}{ab} z_5 z_9 + \frac{\zeta}{ab} z_6 z_8 - 9 z_{10}^2$
- $\frac{\zeta}{b} z_4 z_8 + \frac{1}{b} z_5 z_7 + \frac{\zeta^2}{b} z_6 z_9$
- $\frac{\zeta^2}{b} z_4 z_9 + \frac{\zeta}{ab} z_5 z_8 + \frac{1}{b} z_6 z_7$

Let I be the ideal of $k[z_1, \dots, z_{10}, a, a^{-1}, b, b^{-1}]$ generated by the elements listed in Proposition 2.6. Propositions 2.1 and 2.6 combine to give the following result.

Lemma 2.7. *There is an isomorphism*

$$(k[V] \otimes_k k[\alpha, \alpha^{-1}\beta, \beta^{-1}])^H \cong k[z_1, \dots, z_{10}, a, a^{-1}, b, b^{-1}]/I.$$

Fix $a, b \in k^*$, so that $k[a, a^{-1}, b, b^{-1}] = k$; we are plugging in values for a and b in the ring of Lemma 2.7. Define S to be closed subscheme corresponding to the union of the fixed subspaces of the representation σ of H .

Definition 2.8. *Define an open subset of $\text{Spec } k[z_1, \dots, z_{10}]/I$ by*

$$X(a, b) := \text{Spec } k[z_1, \dots, z_{10}]/I - S.$$

Remark 2.9. Subtracting S causes H to act freely on $X(a, b)$.

By construction, there is an H -torsor $T : W \rightarrow X(a, b)$. Let $q : H \rightarrow \mathbb{Z}/3 \times \mathbb{Z}/3$ be the group quotient homomorphism $q = a_{12} \times a_{23}$ and let $\kappa^{X(a,b)} : \mathcal{O}_{X(a,b)}^* \rightarrow H^1(X(a, b), \mathbb{Z}/3)$ be the Kummer map obtained from $H^*(X(a, b), -)$.

Lemma 2.10. *There is an isomorphism of $\mathbb{Z}/3 \times \mathbb{Z}/3$ -torsors*

$$q_* W \cong \kappa^{X(a,b)}(a) \times \kappa^{X(a,b)}(b).$$

Proof. As a $\mathbb{Z}/3 \times \mathbb{Z}/3$ -torsor, $\kappa^{X(a,b)}(a) \times \kappa^{X(a,b)}(b)$ comes from pulling back

$$\text{Spec } k[z_1, \dots, z_{10}, \gamma, \delta]/(I, \gamma^3 - a, \delta^3 - b) \rightarrow \text{Spec } k[z_1, \dots, z_{10}]/I$$

by the open immersion $X(a, b) \rightarrow \text{Spec } k[z_1, \dots, z_{10}]/I$. Notice that $W = \text{Spec}(k[V] \otimes_k k[\alpha^{\pm 1}, \beta^{\pm 1}]) - T^{-1}(S)$, so $q_* W = (W \times (\mathbb{Z}/3 \times \mathbb{Z}/3))/H$ is an open subset of

$$\text{Spec} \left(\prod_{\mathbb{Z}/3 \times \mathbb{Z}/3} k[V] \otimes_k k[\alpha^{\pm 1}, \beta^{\pm 1}] \right)^H.$$

For any $(m, n) \in \mathbb{Z}/3 \times \mathbb{Z}/3$, we can map

$$k[z_1, \dots, z_{10}, \gamma, \delta]/(I, \gamma^3 - a, \delta^3 - b) \rightarrow k[V] \otimes_k k[\alpha^{\pm 1}, \beta^{\pm 1}]$$

by $\gamma \mapsto \zeta^m \alpha$, $\delta \mapsto \zeta^n \beta$, and by θ (see (2)) on $k[z_1, \dots, z_{10}]$. This gives a map

$$f : k[z_1, \dots, z_{10}, \gamma, \delta] / (I, \gamma^3 - a, \delta^3 - b) \rightarrow \prod_{\mathbb{Z}/3 \times \mathbb{Z}/3} k[V] \otimes_k k[\alpha^{\pm 1}, \beta^{\pm 1}]$$

which takes monomials

$$g(z_1, \dots, z_{10}) \gamma^i \delta^j \mapsto (\theta(g) \zeta^{im+jn} \alpha^i \beta^j)_{(m,n)}.$$

We claim that $\text{Im}(f)$ lies in the H -fixed points. Suppose we have $h \in H$ and a monomial

$$g(z_1, \dots, z_{10}) \gamma^i \delta^j \in k[z_1, \dots, z_{10}, \gamma, \delta] / (I, \gamma^3 - a, \delta^3 - b).$$

Then, using that $\text{Im}(\theta)$ is fixed under the action of H ,

$$h \cdot (\theta(g) \zeta^{im+jn} \alpha^i \beta^j)_{(m,n)} = (\theta(g) \zeta^{i(m+a_{12}(h))+j(n+a_{23}(h))} \alpha^i \beta^j)_{(m+a_{12}(h), n+a_{23}(h))}.$$

This shows that $\text{Im}(f)$ is fixed under the action of H . Hence we get a map on schemes

$$\text{Spec} \left(\prod_{\mathbb{Z}/3 \times \mathbb{Z}/3} k[V] \otimes_k k[\alpha^{\pm 1}, \beta^{\pm 1}] \right)^H \rightarrow \text{Spec} k[z_1, \dots, z_{10}, \gamma, \delta] / (I, \gamma^3 - a, \delta^3 - b).$$

This scheme map is also a map of $\mathbb{Z}/3 \times \mathbb{Z}/3$ -torsors, which we can pullback to a map $\kappa^{X(a,b)}(a) \times \kappa^{X(a,b)}(b) \rightarrow q_* W$ of $\mathbb{Z}/3 \times \mathbb{Z}/3$ -torsors. Since any map of $\mathbb{Z}/3 \times \mathbb{Z}/3$ -torsors is an isomorphism, we are done. \square

We use these torsors to prove a theorem, but we first prove a lemma that will be useful in proving the theorem.

Lemma 2.11. *Let $a, b \in F^*$. If $\tau : \text{Gal } F \rightarrow H$ is such that $q \circ \tau = \kappa(a) \times \kappa(b)$, then $\kappa(a) \smile \kappa(b) = 0$ in $H^2(\text{Spec } F, \mathbb{Z}/3)$.*

Proof. Since $q \circ \tau = \kappa(a) \times \kappa(b)$, we can write

$$(7) \quad \tau(g) = \begin{pmatrix} 1 & \kappa(a)(g) & \tau_{13}(g) \\ 0 & 1 & \kappa(b)(g) \\ 0 & 0 & 1 \end{pmatrix},$$

where $\tau_{13} : H \rightarrow \mathbb{Z}/3$ simply assigns the $(1, 3)$ -entry of $\tau(g)$. Notice that since τ is a homomorphism, we can calculate $\tau_{13}(g_1 g_2)$ for any $g_1, g_2 \in H$ – the $(1, 3)$ -entry of $\tau(g_1) \tau(g_2)$ is

$$(8) \quad \tau_{13}(g_1 g_2) = \tau_{13}(g_1) + \tau_{13}(g_2) + \kappa(a)(g_1) \kappa(b)(g_2).$$

For $d : C^1(H, \mathbb{Z}/3) \rightarrow C^2(H, \mathbb{Z}/3)$ the differential map, we claim that $d(\tau_{13}) = -\kappa(a) \smile \kappa(b)$, whence $\kappa(a) \smile \kappa(b) = 0$ in $H^2(\text{Spec } F, \mathbb{Z}/3)$. By (8), we have

$$\begin{aligned} d(\tau_{13})(g_1, g_2) &= g_1 \tau(g_2) - \tau_{13}(g_1 g_2) + \tau_{13}(g_1) \\ &= g_1 \tau_{13}(g_2) - \tau_{13}(g_2) - \kappa(a)(g_1) \kappa(b)(g_2). \end{aligned}$$

But the action of $\text{Gal } F$ on $\mathbb{Z}/3$ is trivial; hence

$$d(\tau_{13})(g_1, g_2) = -\kappa(a)(g_1) \kappa(b)(g_2) = -(\kappa(a) \smile \kappa(b))(g_1, g_2)$$

as desired. \square

Theorem 2.12. For $a, b \in F^*$, the scheme $X(a, b)$ has an F -point if and only if $\kappa(a) \smile \kappa(b) = 0$ in $H^2(\text{Spec } F, \mathbb{Z}/3)$.

This says that $X(a, b)$ is a splitting variety for $\kappa(a) \smile \kappa(b)$. One direction of this proof is nearly identical to [HW15], and I thank Kirsten Wickelgren for going over that proof with me.

Proof. Suppose there is an F -point $f : \text{Spec } F \rightarrow X(a, b)$. Then the pullback f^*W is an H -torsor. Using Lemma 2.10 and the fact that pushforward and pullback commute,

$$q_* f^* W = f^*(\kappa^{X(a,b)}(a) \times \kappa^{X(a,b)}(b)) = \kappa(a) \times \kappa(b),$$

where the last equality follows from naturality of the Kummer map. Viewing f^*W as an element of $H^1(\text{Gal } F, H)$, let $\tau : \text{Gal } F \rightarrow H$ be a representative of f^*W . Then $q_* f^* W = \kappa(a) \times \kappa(b)$ viewed as an element of $H^1(\text{Gal } F, \mathbb{Z}/3 \times \mathbb{Z}/3)$ has representative $q \circ \tau : \text{Gal } F \rightarrow \mathbb{Z}/3 \times \mathbb{Z}/3$ which takes $g \mapsto (\kappa(a)(g), \kappa(b)(g))$, and this is the only representative. Thus by Lemma 2.11, we find that $\kappa(a) \smile \kappa(b) = 0$.

Conversely, suppose $\kappa(a) \smile \kappa(b) = 0$. Then by [Dwy75, Thm 2.4] there is an element of $H^1(\text{Spec } F, H)$ with representative $\sigma : \text{Gal}(F) \rightarrow H$ such that

$$q_* \sigma = \kappa(a) \times \kappa(b).$$

Take any cube roots α and β of a and b , respectively, in \bar{F} . Then for any $g \in \text{Gal } F$,

$$a_{12}\sigma(g) = \frac{g(\alpha)}{\alpha} \in \mathbb{Z}/3.$$

Similarly, $a_{23}\sigma(g) = g(\beta)/\beta$.

Let $\eta : H \rightarrow \text{GL}_3(F)$ be the homomorphism of the representation $\text{Ind}_N^H \rho$ of H defined earlier. Since $H^1(\text{Spec } F, \text{GL}_3(\bar{F}))$ is the pointed set with one element, the image of σ under $\eta_* : H^1(\text{Spec } F, H) \rightarrow H^1(\text{Spec } F, \text{GL}_3(\bar{F}))$ is trivial. Hence there is $A \in \text{GL}_3(\bar{F})$ such that

$$\eta\sigma(g) = A^{-1}(gA)$$

for all $g \in \text{Gal } F$.

Let $\pi_i : \bar{F}^3 \rightarrow \bar{F}$ be projection onto the i th coordinate. Consider the linear maps $A_i : F^3 \rightarrow \bar{F}$ which take $\mu \mapsto \pi_i(A^{-1}\mu)$. Note that $\dim \text{Ker}(A_i) < 3$. Since F is infinite, F^3 is not contained in a finite union of smaller dimensional vector spaces. Hence there exists $\mu \in F^3$ such that $A^{-1}\mu \in (\bar{F}^*)^3$.

Because $A^{-1}\mu \in (\bar{F}^*)^3$, $A^{-1}\mu \times (\alpha, \beta)$ determines an \bar{F} -point of $\mathbb{G}_m^3 \times \text{Spec } k[\alpha, \alpha^{-1}, \beta, \beta^{-1}]$ by construction. For $g \in \text{Gal } F$, notice that $g^{-1}(A^{-1}(gA)) = (g^{-1}A^{-1})A$. Hence

$$(g^{-1}A^{-1})A = g^{-1}(A^{-1}(gA)) = g^{-1}(\eta\sigma(g)) = \eta\sigma(g),$$

where the last step follows because the image of η is actually contained in $GL_3(F)$. Thus

$$\begin{aligned} g^{-1}(A^{-1}\mu \times (\alpha, \beta)) &= \eta\sigma(g)A^{-1}\mu \times (g^{-1}(\alpha), g^{-1}(\beta)) \\ &= \eta\sigma(g)A^{-1}\mu \times (\alpha\kappa(a)(g^{-1})(\alpha), \beta\kappa(b)(g^{-1})(\beta)) \\ &= \eta\sigma(g)A^{-1}\mu \times (\alpha\kappa(a)(g)(\alpha), \beta\kappa(b)(g)(\beta)) \\ &= \eta\sigma(g)A^{-1}\mu \times q\sigma(g)(\alpha, \beta). \end{aligned}$$

In particular, $A^{-1}\mu \times (\alpha, \beta)$ determines an \bar{F} -point of $X(a, b)$ which by Lemma 2.7 is fixed by the action of $\text{Gal } F$. Hence $A^{-1}\mu \times (\alpha, \beta)$ determines an F point of $X(a, b)$. \square

3. AUTOMATIC REALIZATION OF GALOIS GROUPS

The inverse Galois problem asks, for a given field F and finite group G , whether there is a G -Galois field extension L/F . In the affirmative case, we say that G is *realizable* over L . A related question is that of automatic realization: given two finite groups G and G' , can we decide whether G' is realizable over F solely from the realizability of G over F ? An overview of the theory of automatic realizations can be found in [Jen97]. We prove an automatic realization theorem using the results of Section 2.

Let G be a finite group. A G -Galois ring extension B/A is then the same thing as a G -torsor $\text{Spec } B \rightarrow \text{Spec } A$. We prove a general lemma on the pushforward of such a torsor.

Lemma 3.1. *Let B/A be a finite Galois extension of rings with finite Galois group G and let $q : G \rightarrow Q$ be a quotient of G with kernel K . Then $q_* \text{Spec } B \cong \text{Spec } A^K$.*

Proof. By the definition of pushforward of torsors,

$$q_* \text{Spec } B = \text{Spec} \left(\prod_Q B \right)^G,$$

where G acts on $\prod_Q B$ by

$$g((l_a))_{a \in Q} = (gl_{q(g)^{-1}a})_a.$$

It is enough to show that $(\prod_Q B)^G \cong B^K$. Each element of $Q \cong G/K$ has a representative $g \in G$. We claim that the map $B^K \rightarrow B$ given by $x \mapsto gx$ depends only on $q(g)$. Suppose that $q(g) = q(h)$; then $h^{-1}g \in K$. But then $h^{-1}gx = x$ for all $x \in B^K$, so $gx = hx$. Thus the map induces a ring homomorphism $\varphi : B^K \rightarrow \prod_Q B$ defined by

$$\varphi(l) = (gl)_{q(g) \in Q}.$$

Note that φ is injective because $gx = 0$ if and only if $x = 0$. It remains to show that φ is surjective.

Let $l \in \prod_Q B$. Then

$$h\varphi(l) = (hl_{q(h^{-1}g)})_{q(g)}$$

for any $h \in G$. But there is some $g' \in G$ and $k \in K$ such that $h^{-1}g = g'k$. Hence

$$h\varphi(l) = (hg'l)_{q(g)} = (gl)_{q(g)},$$

so $\text{Im}(\varphi) \subset (\prod_Q B)^G$.

Now suppose that $(l_{q(g)})_{q(g) \in Q}$ is fixed by G . For any $h \in G$,

$$h^{-1}(l_{q(g)})_{q(g)} = (h^{-1}l_{q(h^{-1}g)})_{q(g)};$$

in particular, we must have that $h^{-1}l_{q(h)} = l_{q(1)}$. Moreover, $l_{q(1)} \in B^K$; for if $k \in K$, then $k(l_{q(g)})_{q(g)} = (kl_{q(g)})_{q(g)}$. Therefore $(l_{q(g)})_{q(g)} = \varphi(l_{q(1)})$. \square

We now put this in the context of the rest of the paper. Let H be the mod 3 Heisenberg group and $q : H \rightarrow \mathbb{Z}/3 \times \mathbb{Z}/3$ be the quotient homomorphism. Let F be a number field containing a cube root of unity. Let $X(a, b)$ be as in Definition 2.8.

Theorem 3.2. *Suppose $a, b \in F^*$ are such that $F(\sqrt[3]{a}, \sqrt[3]{b})/F$ is a $\mathbb{Z}/3 \times \mathbb{Z}/3$ -Galois extension. Then the following are equivalent:*

- (1) *There exists a $\mathbb{Z}/3$ -Galois extension $L/F(\sqrt[3]{a}, \sqrt[3]{b})$ such that L/F is an H -Galois extension;*
- (2) *$\kappa(a) \smile \kappa(b) = 0$ in $H^2(\text{Spec } F, \mathbb{Z}/3)$;*
- (3) *The scheme $X(a, b)$ has an F -point.*

Proof. (1) \Rightarrow (2): Suppose that there exists a Galois $\mathbb{Z}/3$ -extension $L/F(\sqrt[3]{a}, \sqrt[3]{b})$ such that L/F is a Galois H -extension. Then $\text{Spec } L \rightarrow \text{Spec } F$ is an H -torsor and $\text{Spec } F(\sqrt[3]{a}, \sqrt[3]{b}) \rightarrow \text{Spec } F$ is a $\mathbb{Z}/3 \times \mathbb{Z}/3$ -torsor. Let $\sigma : \text{Gal } F \rightarrow H$ be a representative of $\text{Spec } L \rightarrow \text{Spec } F$ viewed as an element of $H^1(\text{Spec } F, H)$. Observe $\kappa(a) \times \kappa(b) : \text{Gal } F \rightarrow \mathbb{Z}/3 \times \mathbb{Z}/3$ is the unique cocycle representative of $\text{Spec } F(\sqrt[3]{a}, \sqrt[3]{b}) \rightarrow \text{Spec } F$. By Lemma 3.1, we get that

$$q_* \text{Spec } L \cong \text{Spec } L^{\mathbb{Z}/3} = \text{Spec } F(\sqrt[3]{a}, \sqrt[3]{b});$$

hence $q_* \sigma = \kappa(a) \times \kappa(b)$. Applying Lemma 2.11, we conclude that $\kappa(a) \smile \kappa(b) = 0$.

(2) \Rightarrow (3): This is Theorem 2.12.

(3) \Rightarrow (1): Suppose there is an F -point $f : \text{Spec } F \rightarrow X(a, b)$. Then as in the proof of Theorem 2.12, we can pullback f to an H -torsor $f^*W \rightarrow \text{Spec } F$ such that $q_* f^*W = \kappa(a) \times \kappa(b)$. Since $F(\sqrt[3]{a}, \sqrt[3]{b})/F$ is a $\mathbb{Z}/3 \times \mathbb{Z}/3$ -Galois extension, it follows that $q_* f^*W = \text{Spec } F(\sqrt[3]{a}, \sqrt[3]{b})$. Note that $f^*W \rightarrow \text{Spec } F$ is finite by [SGA1, V Prop 2.6 (iii) (i)], so $f^*Y = \text{Spec } L$ for a finitely generated F -module L . Furthermore, L has an action of H and $L^H = F$. Thus it is enough to show that L is a field.

For this, it suffices to show that L is an integral domain, because L is a finitely generated F -module. By Lemma 3.1,

$$L^{\mathbb{Z}/3} = q_* f^*W = F[\sqrt[3]{a}, \sqrt[3]{b}].$$

Hence $\text{Spec } L$ is a $\mathbb{Z}/3$ -torsor over $\text{Spec } F[\sqrt[3]{a}, \sqrt[3]{b}]$. But these are all known: in particular, either L is a field or this is the trivial torsor. We show that $\text{Spec } L \rightarrow \text{Spec } F[\sqrt[3]{a}, \sqrt[3]{b}]$ is not trivial, which completes the proof.

Suppose it were trivial, that is

$$\text{Spec } L \cong \text{Spec } \prod_{\mathbb{Z}/3} F[\sqrt[3]{a}, \sqrt[3]{b}].$$

We show that this implies that $\text{Spec } L \rightarrow \text{Spec } F$ is not an H -torsor, which gives a contradiction. We know that E_{13} acts on L by permuting the factors. We can use this to solve for the actions of E_{12} and E_{23} . For $x \in F[\sqrt[3]{a}, \sqrt[3]{b}]$, the action of E_{12} on (x, x, x) is given by acting

E_{12} on each coordinate. Let $E_{12}(1, 0, 0) = (d, e, f)$. Then

$$E_{12}(0, 1, 0) = E_{13}E_{12}(1, 0, 0) = (f, d, e)$$

because the actions of E_{23} and E_{13} commute on $\prod_{\mathbb{Z}/3} F$. Similarly, $E_{12}(0, 0, 1) = (e, f, d)$.

Let $g = d + e + f$. Then $E_{12}(1, 1, 1) = (g, g, g)$. But H fixes F and E_{12} acts on $(1, 1, 1)$ by acting on each coordinate, so $g = 1$. Now, $(1, 0, 0)(0, 1, 0) = (0, 0, 0)$, so $E_{12}(1, 0, 0)(0, 1, 0) = (0, 0, 0)$. On the other hand,

$$E_{12}(1, 0, 0)(0, 1, 0) = (d, e, f)(f, d, e) = (df, de, ef).$$

Hence two of d, e, f are 0 and the other is 1.

From this and the fact that $E_{12}(x, 0, 0) = E_{12}(1, 0, 0)(x, x, x)$, we find that $E_{12}(x, 0, 0)$ is one of $(E_{12}x, 0, 0)$, $(0, E_{12}x, 0)$, or $(0, 0, E_{12}(x))$. This choice will determine the entire action of E_{12} on L . The same argument works for E_{23} as well.

We have shown that E_{12} and E_{23} act by translating the factors and then acting on each factor. But then it follows that E_{12} and E_{23} commute, which is a contradiction. Therefore $\text{Spec } L \rightarrow \text{Spec } F[\sqrt[3]{a}, \sqrt[3]{b}]$ is not the trivial torsor. \square

REFERENCES

- [Art10] Michael Artin, *Algebra*, Pearson Prentice Hall, 2010.
- [BF03] Grégory Berhuy and Giordano Favi, *Essential dimension: A functorial point of view (after A. Merkurjev)*, Doc. Math. **8** (2003), 279–330.
- [Dwy75] William G. Dwyer, *Homology, Massey products and maps between groups*, J. Pure. Appl. Algebra **6** (1975), 177–190.
- [FH91] William Fulton and Joe Harris, *Representation theory: A first course*, Springer-Verlag, 1991.
- [GMS03] Skip Garabaldi, Alexander Merkurjev, and Jean-Pierre Serre, *Cohomological invariants in Galois cohomology*, University Lecture Series, vol. 28, American Mathematical Society, Providence, RI, 2003.
- [GS] Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at <http://www.math.uiuc.edu/Macaulay2/>.
- [HW15] Michael J. Hopkins and Kirsten G. Wickelgren, *Splitting varieties for triple Massey products*, J. Pure. Appl. Algebra **219** (2015), 1304–1319.
- [Jen97] C.U. Jensen, *Elementary questions in Galois theory*, Algebra Logic Appl. **9** (1997), 11–24.
- [SGA1] *Revêtements étales et groupe fondamental*, Springer-Verlag, Berlin, 1971, Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1), Directed by A. Grothendieck. With two papers by M. Raynaud, Lecture Notes in Mathematics, Vol. 224.
- [Stu96] Bernd Sturmfels, *Gröbner bases and convex polytopes*, American Mathematical Society, 1996.

SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA GA

E-mail address: bboguess3@gatech.edu